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B.Sc.– III

MATHEMATICS HONS: Paper – V

group B (Multiple Integrals)

Contents :→ stoke's theorem.

THEOREM (stoke's Theorem) :→ (Relation between line & surface integrals)

If S be an open surface bounded by a closed curve C and $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be any continuously differentiable vector point function then

$$\int_C \vec{F}\cdot d\vec{R} = \int_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds \qquad \text{——— (1)}$$

Where $\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$ is a unit external normal at any point S.

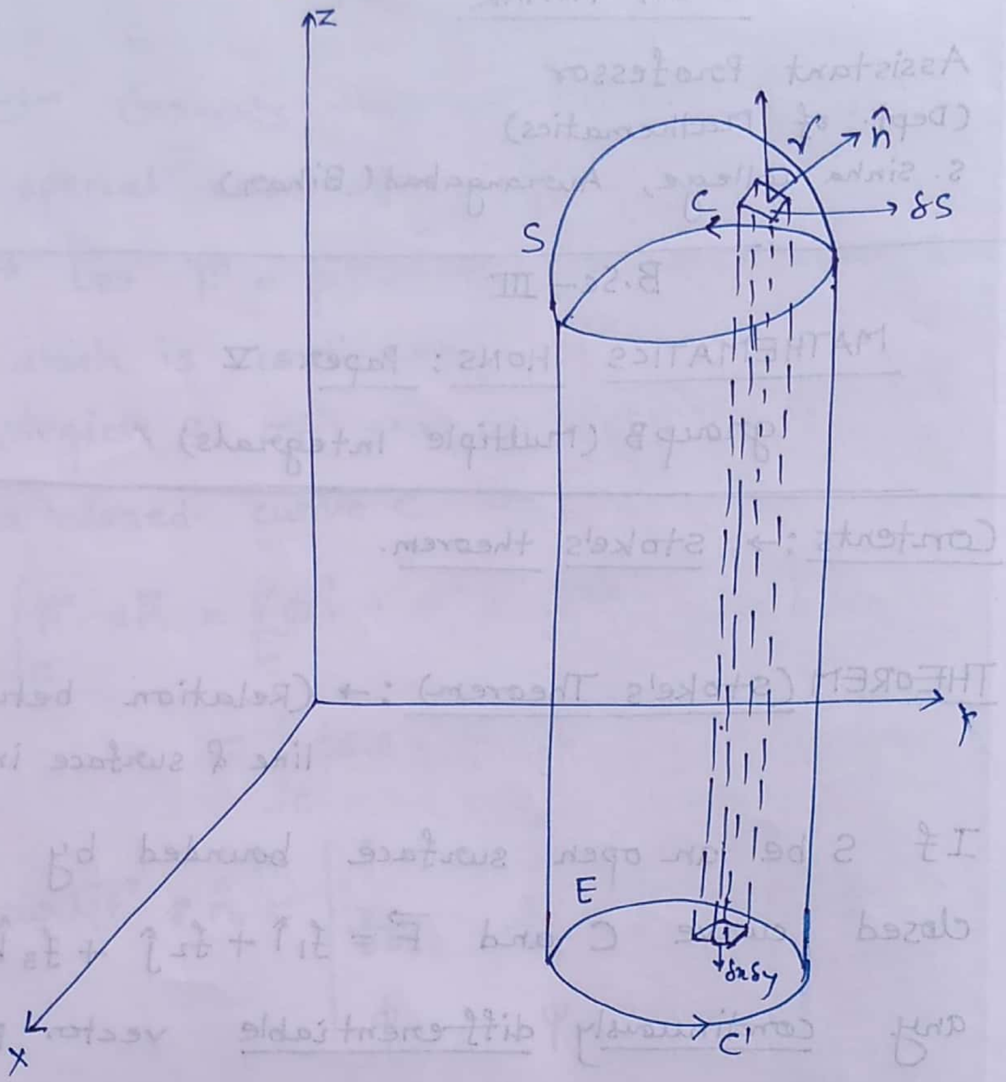
Proof :→ Writing $d\vec{R} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$, then (1) takes the form

$$\int_C f_1dx + f_2dy + f_3dz = \int_S \left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\cos\beta + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\cos\gamma\right]ds \qquad \text{——— (2)}$$

Let us first prove that

$$\int_C f_1dx = \int_S \left(\frac{\partial f_1}{\partial z}\cos\beta - \frac{\partial f_1}{\partial y}\cos\gamma\right)ds \qquad \text{——— (3)}$$

Z.




Let $z = f(x,y)$ be the equation of the surface S, whose projection on the xy-plane is the region E. Then the projection of C on the xy-plane is the curve C' enclosing the region E.

$$\therefore \int_C f_1(x,y,z)\, dx = \int_{C'} f(x, y, g(x,y))\, dx$$

$$= -\iint_E \frac{\partial}{\partial y} f_1(x,y,g)\, dx\, dy \quad \text{(By green's theorem)}$$

$$= -\iint_E \left( \frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial z} \cdot \frac{\partial g}{\partial y} \right) dx\, dy \quad \text{——— (4)}$$

The direction cosines of the normal to the surface $z = g(x,y)$ are given by

$$\frac{\cos\alpha}{-\frac{\partial g}{\partial x}} = \frac{\cos\beta}{-\frac{\partial g}{\partial y}} = \frac{\cos\gamma}{1} \qquad \text{--- (5)}$$

$\therefore$ $dx\,dy$ = projection of $ds$ on the $xy$-plane.

$$= ds\cos\gamma$$

$$\Rightarrow ds = \frac{dx\,dy}{\cos\gamma}$$

$$\therefore \int_S \left(\frac{\partial f_1}{\partial z}\cos\beta - \frac{\partial f_1}{\partial y}\cos\gamma\right) ds$$

$$= \iint_E \left(\frac{\partial f_1}{\partial z}\frac{\cos\beta}{\cos\gamma} - \frac{\partial f_1}{\partial y}\right) dx\,dy$$

$$= -\iint_E \left(\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial z}\cdot\frac{\partial g}{\partial y}\right) dx\,dy \qquad \left[\because \frac{\cos\beta}{\cos\gamma} = -\frac{\partial g}{\partial y} \text{ from (5)}\right]$$

$$\therefore \int_C f_1\,dx = \int_S \left(\frac{\partial f_1}{\partial z}\cos\beta - \frac{\partial f_1}{\partial y}\cos\gamma\right) dx\,dy \qquad \text{--- (6)}$$

similarly we can prove.

$$\int_C f_2\,dy = \int_S \left(\frac{\partial f_2}{\partial x}\cos\gamma - \frac{\partial f_2}{\partial z}\cos\alpha\right) ds \qquad \text{(7)}$$

$$\int_C f_3\,dz = \int_S \left(\frac{\partial f_3}{\partial y}\cos\alpha - \frac{\partial f_3}{\partial x}\cos\beta\right) ds \qquad \text{--- (8)}$$

Adding (6), (7) & (8), we get

$$\boxed{\int_C f_1\,dx + f_2\,dy + f_3\,dz = \int_S \operatorname{curl}\vec{F}\cdot\hat{n}\,ds}$$

proved.

Cor:→ Green's theorem in a plane as a special case of stokes theorem.

→ Let $\vec{F} = \phi\hat{i} + \psi\hat{j}$ be a vector function which is continuously differentiable in a region S of the xy- plane bounded by a closed curve C. Then

$$\int_C \vec{F}\cdot d\vec{R} = \int_C (\phi\hat{i} + \psi\hat{j})\cdot(dx\hat{i} + dy\hat{j})$$

$$= \int_C \phi dx + \psi dy$$

$$\text{curl}\,\vec{F}\cdot\hat{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi & \psi & 0 \end{vmatrix}\cdot\hat{k}$$

$$= \frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}$$

Hence stoke's theorem takes the form

$$\int_C \phi dx + \psi dy = \int_C \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dx\,dy$$

Which is greens theorem.